GS [125 marks]

1. [Maximum mark: 8]

Consider a sequence of ten rectangular picture frames $F_1, F_2, \ldots, F_9, F_{10}$.

Picture frame F_1 has width $4\ \mathrm{cm}$ and height $5\ \mathrm{cm}$.

The width and height of picture frame F_n, are each increased by $50\,\%$ to generate the width and height of the next picture frame F_{n+1}, for $n \in \mathbb{Z}^+$, $1 \leq n \leq 9$.

(a.i) Show that the area of picture frame F_n is $20\left(\frac{9}{4}\right)^{n-1}\ \mathrm{cm}^2$. [2]

(a.ii) Hence, find the mean area of the ten picture frames, giving your answer in the form $p\left(\left(\frac{9}{4}\right)^a - 1\right)\ \mathrm{cm}^2$, where $p \in \mathbb{Q}^+$, $a \in \mathbb{Z}^+$. [3]

(b) Find the median area of the ten picture frames, giving your answer in the form $q\left(\frac{9}{4}\right)^4\ \mathrm{cm}^2$, where $q \in \mathbb{Q}^+$. [3]

2. [Maximum mark: 16]

Consider the sequence $\{u_n\}$, with $n\,\mathrm{th}$ term given by u_n. The first three terms are

$$u_1 = k - 5,\ u_2 = 3 - 2k \text{ and } u_3 = 5k + 3, \text{ where } k \in \mathbb{R}.$$

(a) Consider the case when $\{u_n\}$ is arithmetic.

(a.i) Find the value of k. [3]

(a.ii) Hence, or otherwise, find u_3. [2]

(b) Consider the case where $k = 12$.

(b.i) Show that the first three terms of $\{u_n\}$ form a geometric sequence. [3]

(b.ii) Given that $\{u_n\}$ is geometric, state a reason why the sum of an infinite number of terms of this sequence does not exist. [1]

(c) The sequence, $\{u_n\}$, is geometric for a second value of k.

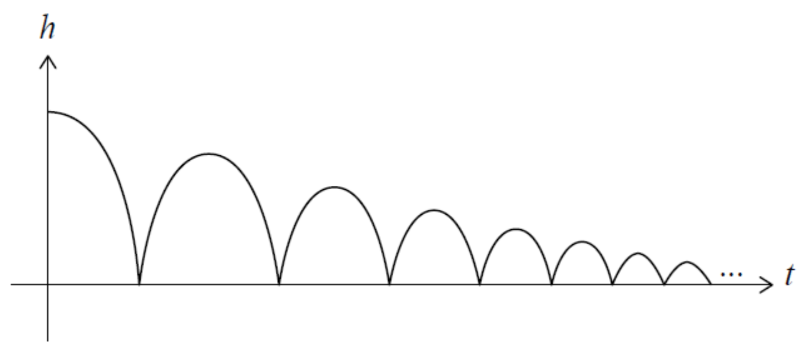
(c.i) Show that $k^2 - 10k - 24 = 0$. [2]

(c.ii) Find the first three terms of $\{u_n\}$ for this second value of k. [4]

(c.iii) Hence, write down the value of S_{2m}, the sum of the first $2m$ terms, for this second value of k. [1]

3. [Maximum mark: 14]

A tennis ball is dropped from a height. After each bounce the maximum height reached by the ball is $\frac{2}{3}$ of its previous maximum height. This can be seen in the diagram below where h, in metres, is the height of a ball after t seconds.



A box contains tennis balls. Each ball satisfies the condition of rebounding to $\frac{2}{3}$ of their previous maximum height. The tennis balls are numbered Ball $1,\ 2,\ 3\ldots$.

Ball 1 is dropped from a height of 10 metres.

(a) Find the maximum height of Ball 1 after the 5^{th} bounce. [3]

(b) Find the total distance travelled by Ball 1 immediately before the 5^{th} bounce. [3]

Let δ be the total distance travelled by any of these balls.

(c) A ball is dropped from a height of x metres. Show that $\delta = 5x$ metres. [3]

Let δ_1 be the total distance travelled by Ball 1.

(d) Write down the value of δ_1. [1]

Ball 2 is dropped from a height of 9.56 metres.

Let δ_2 be the total distance travelled by Ball 2, and so on for each ball in the box.

It is given that $\delta_1,\ \delta_2\ ,\ \delta_3\ \ldots$ form an arithmetic sequence.

(e) Determine which tennis ball is the first ball to travel less than 25 metres. [4]

4. [Maximum mark: 5]

Consider a geometric sequence with first term 1 and common ratio 10.

S_n is the sum of the first n terms of the sequence.

(a) Find an expression for S_n in the form $\frac{a^n-1}{b}$, where $a,\ b \in \mathbb{Z}^+$. [1]

(b) Hence, show that $S_1 + S_2 + S_3 + \ldots + S_n = \frac{10(10^n-1)-9n}{81}$. [4]

5. [Maximum mark: 16]

Consider the arithmetic sequence $a,\ p,\ q\ldots$, where $a,\ p,\ q \neq 0$.

(a) Show that $2p - q = a$. [2]

Consider the geometric sequence $a,\ s,\ t\ldots$, where $a,\ s,\ t \neq 0$.

(b) Show that $s^2 = at$. [2]

The first term of both sequences is a.

It is given that $q = t = 1$.

(c) Show that $p > \frac{1}{2}$. [2]

Consider the case where $a = 9,\ s > 0$ and $q = t = 1$.

(d) Write down the first four terms of the

(d.i) arithmetic sequence; [2]

(d.ii) geometric sequence. [2]

The arithmetic and the geometric sequence are used to form a new arithmetic sequence u_n.

The first three terms of u_n are $u_1 = 9 + \ln 9,\ u_2 = 5 + \ln 3$, and $u_3 = 1 + \ln 1$.

(e.i) Find the common difference of the new sequence in terms of $\ln 3$. [3]

(e.ii) Show that $\sum_{i=1}^{10} = -90 - 25 \ln 3$. [3]

6. [Maximum mark: 7]

Darren buys a car for $\$35\,000$. The value of the car decreases by $15\,\%$ in the first year.

(a) Find the value of the car at the end of the first year. [2]

After the first year, the value of the car decreases by $11\,\%$ in each subsequent year.

(b) Find the value of Darren's car 10 years after he buys it, giving your answer to the nearest dollar. [2]

When Darren has owned the car for n complete years, the value of the car is less than $10\,\%$ of its original value.

(c) Find the least value of n. [3]

7. [Maximum mark: 14]

Consider the arithmetic sequence $u_1, u_2, u_3, \ldots$.

The sum of the first n terms of this sequence is given by $S_n = n^2 + 4n$.

(a.i) Find the sum of the first five terms. [2]

(a.ii) Given that $S_6 = 60$, find u_6. [2]

(b) Find u_1. [2]

(c) Hence or otherwise, write an expression for u_n in terms of n. [3]

Consider a geometric sequence, v_n, where $v_2 = u_1$ and $v_4 = u_6$.

(d) Find the possible values of the common ratio, r. [3]

(e) Given that $v_{99} < 0$, find v_5. [2]

8. [Maximum mark: 15]

Consider the series $\ln x + p \ln x + \frac{1}{3} \ln x + \ldots$, where $x \in \mathbb{R},\ x > 1$ and $p \in \mathbb{R},\ p \neq 0$.

Consider the case where the series is geometric.

(a.i) Show that $p = \pm\frac{1}{\sqrt{3}}$. [2]

(a.ii) Given that $p > 0$ and $S_\infty = 3 + \sqrt{3}$, find the value of x. [3]

Now consider the case where the series is arithmetic with common difference d.

(b.i) Show that $p = \frac{2}{3}$. [3]

(b.ii) Write down d in the form $k \ln x$, where $k \in \mathbb{Q}$. [1]

(b.iii) The sum of the first n terms of the series is $-3 \ln x$.

Find the value of n. [6]

9. [Maximum mark: 15]

Consider the function $f(x) = a^x$ where $x,\ a \in \mathbb{R}$ and $x > 0,\ a > 1$.

The graph of f contains the point $\left(\frac{2}{3}, 4\right)$.

(a) Show that $a = 8$. [2]

(b) Write down an expression for $f^{-1}(x)$. [1]

(c) Find the value of $f^{-1}\left(\sqrt{32}\right)$. [3]

Consider the arithmetic sequence

$\log_8 27\ ,\ \ \log_8 p\ ,\ \ \log_8 q\ ,\ \ \log_8 125$, where $p > 1$ and $q > 1$.

(d.i) Show that $27,\ p,\ q$ and 125 are four consecutive terms in a geometric sequence. [4]

(d.ii) Find the value of p and the value of q. [5]

10. [Maximum mark: 9]

The sum of the first n terms of a geometric sequence is given by

$S_n = \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r.$

(a) Find the first term of the sequence, u_1. [2]

(b) Find S_∞. [3]

(c) Find the least value of n such that $S_\infty - S_n < 0.001$. [4]

11. [Maximum mark: 6]

Mia baked a very large apple pie that she cuts into slices to share with her friends. The smallest slice is cut first. The volume of each successive slice of pie forms a geometric sequence.

The second smallest slice has a volume of $30\,\mathrm{cm}^3$. The fifth smallest slice has a volume of $240\,\mathrm{cm}^3$.

(a) Find the common ratio of the sequence. [2]

(b) Find the volume of the smallest slice of pie. [2]

(c) The apple pie has a volume of $61\,425\,\mathrm{cm}^3$.

Find the total number of slices Mia can cut from this pie. [2]